PROBLEMS FOR SEPTEMBER

Please send your solutions to
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no later than **October 31, 2001**.

Notes. A unit cube (tetrahedron) is a cube (tetrahedron) all of whose side lengths are 1.

90. Let m be a positive integer, and let $f(m)$ be the smallest value of n for which the following statement is true:

given any set of n integers, it is always possible to find a subset of m integers whose sum is divisible by m

Determine $f(m)$.

[*Comment.* This problem is being reposed, as no one submitted a complete solution to this problem the first time around. Can you conjecture what $f(m)$ is? It is not hard to give a lower bound for this function. One approach is to try to relate $f(a)$ and $f(b)$ to $f(ab)$ and reduce the problem to considering the case that m is prime; this give access to some structure that might help.]

103. Determine a value of the parameter θ so that

$$f(x) \equiv \cos^2 x + \cos^2(x+\theta) - \cos x \cos(x+\theta)$$

is a constant function of x.

104. Prove that there exists exactly one sequence $\{x_n\}$ of positive integers for which

$$x_1 = 1 \ , \quad x_2 > 1 \ , \qquad x_{n+1}^3 + 1 = x_n x_{n+2}$$

for $n \geq 1$.

105. Prove that within a unit cube, one can place two regular unit tetrahedra that have no common point.

106. Find all pairs (x, y) of positive real numbers for which the least value of the function

$$f(x,y) = \frac{x^4}{y^4} + \frac{y^4}{x^4} - \frac{x^2}{y^2} - \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x}$$

is attained. Determine that minimum value.

107. Given positive numbers a_i with $a_1 < a_2 < \cdots < a_n$, for which permutation $(b_1, b_2, \cdots, b_n)$ of these numbers is the product

$$\prod_{i=1}^{n} \left(a_i + \frac{1}{b_i} \right)$$

maximized?

108. Determine all real-valued functions $f(x)$ of a real variable x for which

$$f(xy) = \frac{f(x) + f(y)}{x+y}$$

for all real x and y for which $x + y \neq 0$.