PROBLEMS FOR MARCH

Please send your solutions to
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no later than **April 30, 2001**.

Notes. A real-valued function f defined on an interval is *concave* iff $f((1-t)u+tv) \geq (1-t)f(u)+tf(v)$ whenever $0 < t < 1$ and u and v are in the domain of definition of $f(x)$. If $f(x)$ is a one-one function defined on a domain into a range, then the *inverse* function $g(x)$ defined on the set of values assumed by f is determined by $g(f(x)) = x$ and $f(g(y)) = y$; in other words, $f(x) = y$ if and only if $g(y) = x$.

67. (a) Consider the infinite integer lattice in the plane (*i.e.*, the set of points with integer coordinates) as a graph, with the edges being the lines of unit length connecting nearby points. What is the minimum number of colours that can be used to colour all the vertices and edges of this graph, so that

    (i) each pair of adjacent vertices gets two distinct colours; AND

    (ii) each pair of edges that meet at a vertex get two distinct colours; AND

    (iii) an edge is coloured differently that either of the two vertices at the ends?

    (b) Extend this result to lattices in real $n-$dimensional space.

68. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

69. Let n, a_1, a_2, $\cdots$, a_k be positive integers for which $n \geq a_1 > a_2 > a_3 > \cdots > a_k$ and the least common multiple of a_i and a_j does not exceed n for all i and j. Prove that $ia_i \leq n$ for $i = 1, 2, \cdots, k$.

70. Let $f(x)$ be a concave strictly increasing function defined for $0 \leq x \leq 1$ such that $f(0) = 0$ and $f(1) = 1$. Suppose that $g(x)$ is its inverse. Prove that $f(x)g(x) \leq x^2$ for $0 \leq x \leq 1$.

71. Suppose that lengths a, b and i are given. Construct a triangle ABC for which $|AC| = b$. $|AB| = c$ and the length of the bisector AD of angle A is i (D being the point where the bisector meets the side BC).

72. The centres of the circumscribed and the inscribed spheres of a given tetrahedron coincide. Prove that the four triangular faces of the tetrahedron are congruent.