PROBLEMS FOR AUGUST

Please send your solutions to
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no later that **September 30, 2001**. Unless you are submitting in TeX, please do not submit your solutions as an electronic attachment. Make sure that your name, complete mailing address and email address are on the front page of your solution set.

Notes. A *rectangular hyperbola* is an hyperbola whose asymmptotes are at right angles.

97. A triangle has its three vertices on a rectangular hyperbola. Prove that its orthocentre also lies on the hyperbola.

98. Let $a_1, a_2, \cdots, a_{n+1}, b_1, b_2, \cdots, b_n$ be nonnegative real numbers for which
(i) $a_1 \geq a_2 \geq \cdots \geq a_{n+1} = 0$,
(ii) $0 \leq b_k \leq 1$ for $k = 1, 2, \cdots, n$.
Suppose that $m = \lfloor b_1 + b_2 + \cdots + b_n \rfloor + 1$. Prove that

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{m} a_k \ .$$

99. Let E and F be respective points on sides AB and BC of a triangle ABC for which $AE = CF$. The circle passing through the points B, C, E and the circle passing through the points A, B, F intersect at B and D. Prove that BD is the bisector of angle ABC.

100. If 10 equally spaced points around a circle are joined consecutively, a convex regular inscribed decagon P is obtained; if every third point is joined, a self-intersecting regular decagon Q is formed. Prove that the difference between the length of a side of Q and the length of a side of P is equal to the radius of the circle. [*With thanks to Ross Honsberger.*]

101. Let a, b, u, v be nonnegative. Suppose that $a^5 + b^5 \leq 1$ and $u^5 + v^5 \leq 1$. Prove that

$$a^2u^3 + b^2v^3 \leq 1 \ .$$

[*With thanks to Ross Honsberger.*]

102. Prove that there exists a tetrahedron $ABCD$, all of whose faces are similar right triangles, each face having acute angles at A and B. Determine which of the edges of the tetrahedron is largest and which is smallest, and find the ratio of their lengths.