Solutions and Comments

43. Two players play a game: the first player thinks of n integers x_1, x_2, $\cdots$, x_n, each with one digit, and the second player selects some numbers a_1, a_2, $\cdots$, a_n and asks what is the value of the sum $a_1x_1 + a_2x_2 + \cdots + a_nx_n$. What is the minimum number of questions used by the second player to find the integers x_1, x_2, $\cdots$, x_n?

Solution. We are going to prove that the second player needs only one question to find the integers x_1, x_2, $\cdots$, x_n. Indeed, let him choose $a_1 = 100$, $a_2 = 100^2$, $\cdots$, $a_n = 100^n$ and ask for the value of the sum

$$S_n = 100x_1 + 100^2x_2 + \cdots + 100^nx_n \quad .$$

Note that

$$\begin{aligned}
&\left|\frac{100x_1 + 100^2x_2 + \cdots + 100^{n-1}x_{n-1}}{100^n}\right| \\
&\quad\le \frac{100|x_1| + 100^2|x_2| + \cdots + 100^{n-1}|x_{n-1}|}{100^n} \\
&\quad\le \frac{9(100 + 100^2 + \cdots + 100^{n-1})}{100^n} \\
&\quad< \frac{10^{2n-1}}{10^{2n}} = \frac{1}{10} \ .
\end{aligned}$$

Hence

$$\left|\frac{S_n}{100^n} - x_n\right| = \frac{100x_1 + 100^2x_2 + \cdots + 100^{n-1}x_{n-1}}{100^n} < \frac{1}{10} \ ,$$

and x_n can be obtained. Now, we can find the sum

$$S_{n-1} = S_n - 100^nx_n = 100x_1 + 100^2x_2 + \cdots + 100^{n-1}x_{n-1}$$

and similarly obtain x_{n-1}. The procedure continues until all the numbers are found.

44. Find the permutation $\{a_1, a_2, \cdots, a_n\}$ of the set $\{1, 2, \cdots, n\}$ for which the sum

$$S = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}|$$

has maximum value.

Solution. Let $a = (a_1, a_2, \cdots, a_n)$ be a permutation of $\{1, 2, \cdots, n\}$ and define

$$f(a) = \sum_{k=1}^{n-1} |a_{k+1} - a_k| + |a_n - a_1| \ .$$

With $a_{n+1} = a_1$ and $\epsilon_0 = \epsilon_n$, we find that

$$f(a) = \sum_{k=1}^{n} \epsilon_k(a_{k+1} - a_k) = \sum_{k=1}^{n} (\epsilon_{k-1} - \epsilon_k)a_k$$

where ϵ_1, ϵ_2, $\cdots$, ϵ_n are all equal to 1 or -1. Thus

$$f(a) = \sum_{k=1}^{n} \beta_k k \ .$$

where each β_k is one of -2, 0, 2, and $\sum_{k=1}^{n} \beta_k = 0$ (there are the same number of positive and negative numbers among the β_k).

Therefore

$$f(a) = 2(x_1 + x_2 + \cdots + x_m) - 2(y_1 + y_2 + \cdots + y_m)$$

where $x_1, \cdots, x_m, y_1, \cdots, y_m \in \{1, 2, \cdots, n\}$ and are distinct from each other. Hence $f(a)$ is maximum when $\{x_1, x_2, \cdots, x_m\} = \{n, n-1, \cdots, n-m+1\}$, $\{y_1, y_2, \cdots, y_m\} = \{1, 2, \cdots, m\}$ with $m = \lfloor n/2 \rfloor$, *i.e.*, m is as large as possible. The maximum value is

$$2\left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) .$$

This value is attained by taking

$$a = (s+1, 1, s+2, 2, \cdots, 2s, s) \qquad \text{when} \;\; n = 2s$$

and

$$a = (s+2, 1, s+3, 2, \cdots, 2s+1, s, s+1) \qquad \text{when } n = 2s+1 .$$

Since $|a_n - a_1| = 1$ for these permutations, the maximum value of the given expression is

$$2\left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) - 1 .$$

This is equal to $2s^2 - 1$ when $n = 2s$, and to $2s(s+1) - 1$ when $n = 2s+1$.

45. Prove that there is no nonconstant polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with integer coefficients a_i for which $p(m)$ is a prime number for every integer m.

Solution. Let a be an integer, for which $p(a) \neq -1, 0, 1$. (If there is no such a, then p cannot take all prime values.) Suppose that b is a prime divisor of $p(a)$. Now, for any integer k,

$$p(a+kb) - p(a) = a_n[(a+kb)^n - a^n] + a_{n-1}[(a+kb)^{n-1} - a_{n-1}] + \cdots + a_1[(a+kb) - a] .$$

It can be seen that b is a divisor of $p(a+kb) - p(a)$ and hence of $p(a+kb)$ for every integer k. Both of the equations $p(x) = b$ and $p(x) = -b$ have at most finitely many roots. So some of the values of $p(a+kb)$ must be composite, and the result follows.

Comment. It should have been stated in the problem that the polynomial was nonconstant, or had positive degree.

46. Let $a_1 = 2$, $a_{n+1} = \frac{a_n + 2}{1 - 2a_n}$ for $n = 1, 2, \cdots$. Prove that

(a) $a_n \neq 0$ for each positive integer n;

(b) there is no integer $p \geq 1$ for which $a_{n+p} = a_n$ for every integer $n \geq 1$ (*i.e.*, the sequence is not periodic).

Solution. (a) We prove that $a_n = \tan n\alpha$ where $\alpha = \arctan 2$ by mathematical induction. This is true for $n = 1$. Assume that it holds for $n = k$. Then

$$a_{k+1} = \frac{2 + a_n}{1 - 2a_n} = \frac{\tan\alpha + \tan n\alpha}{1 - \tan\alpha \tan n\alpha} = \tan(n+1)\alpha ,$$

as desired.

Suppose that $a_n = 0$ with $n = 2m+1$. Then $a_{2m} = -2$. However,

$$a_{2m} = \tan 2m\alpha = \frac{2\tan m\alpha}{1 - \tan^2 m\alpha} = \frac{2a_m}{1 - a_m^2} ,$$

whence

$$\frac{2a_m}{1-a_m^2} = -2 \Leftrightarrow a_m = \frac{1 \pm \sqrt{5}}{2} ,$$

which is not possible, since a_m has to be rational.

Suppose that $a_n = 0$ with $n = 2^k(2m+1)$ for some positive integer k. Then

$$0 = a_n = \tan 2 \cdot 2^{k-1}(2m+1)\alpha = \frac{2 \tan 2^{k-1}(2m+1)}{1 - \tan^2 2^{k-1}(2m+1)} = \frac{2a_{n/2}}{1-a_{n/2}^2} ,$$

so that $a_{n/2} = 0$. Continuing step by step backward, we finally come to $a_{2m+1} = 0$, which has already been shown as impossible.

(b) Assume, if possible, that the sequence is periodic, *i.e.*, there is a positive integer p such that $a_{n+p} = a_n$ for every positive integer n. Thus

$$\tan(n+p)\alpha - \tan n\alpha = \frac{\sin p\alpha}{\cos(n+p)\alpha \cos n\alpha} = 0 .$$

Therefore $\sin p\alpha = 0$ and so $a_p = \tan p\alpha = 0$, which, as we have shown, is impossible. The desired result follows.

47. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{s - a_k} \le 1$$

where $s = 1 + a_1 + a_2 + \cdots + a_n$.

Solution. First, we recall that Chebyshev's Inequalities:
(1) if the vectors $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are similarly sorted (that is, both rising or both falling), then

$$\frac{a_1 b_1 + \cdots + a_n b_n}{n} \ge \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} ;$$

(2) if the vectors $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are oppositely sorted (that is, one rising and the other falling), then

$$\frac{a_1 b_1 + \cdots + a_n b_n}{n} \le \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} .$$

If $x_1, x_2, \cdots, x_n$ are positive real numbers with $x_1 \le x_1 \le \cdots \le x_n$, then $x_1^n \le x_2^n \le \cdots \le x_n^n$. From Chebyshev's Inequality (1), we have, for each $k = 1, 2, \cdots, n$, that

$$\sum_{i=1, i\neq k}^{n} x_i^n = \sum_{i=1, i\neq k}^{n} x_i^{n-1} x_i \ge \frac{1}{n-1} \Bigg(\sum_{i=1, i\neq k}^{n} x_i \Bigg) \Bigg(\sum_{i=1, i\neq k}^{n} x_i^{n-1} \Bigg) .$$

The Arithmetic-Geometric Means Inequality yields

$$\sum_{i=1, i\neq k}^{n} x_i^{n-1} \ge (n-1) x_1 \cdots x_{k-1} x_{k+1} \cdots x_n ,$$

for $k = 1, \cdots, n$. Therefore,

$$\sum_{i=1, i\neq k}^{n} x_i^n \ge (x_1 \cdots x_{k-1} x_{k+1} \cdots x_n) \sum_{i=1, i\neq k}^{n} x_i .$$

for each k. This inequality can also be written

$$x_1^n + \cdots + x_{k-1}^n + x_{k+1}^n + \cdots x_n^n + x_1 x_2 \cdots x_n$$

$$\geq x_1 \cdots x_{k-1} x_{k+1} \cdots x_n (x_1 + x_2 + \cdots + x_n) \ ,$$

or

$$\frac{1}{x_1 x_2 \cdots x_n + x_1^n + \cdots + x_{k-1}^n + x_{k+1}^n + \cdots + x_n^n} \leq \frac{1}{x_1 + x_2 + \cdots + x_n} \cdot \frac{1}{x_1 \cdots x_{k-1} x_{k+1} \cdots x_n} \ .$$

Adding up these inequalities, for $1 \leq k \leq n$, we get

$$\sum_{k=1}^{n} \frac{1}{x_1 x_2 \cdots x_n + x_1^n + \cdots + x_{k-1}^n + x_{k+1}^n + \ldots x_n^n} \leq \frac{1}{x_1 x_2 \cdots x_n} \ .$$

Now, let the x_k^n be equal to the a_k in increasing order to obtain the desired result.

48. Let $A_1 A_2 \cdots A_n$ be a regular $n-$gon and d an arbitrary line. The parallels through A_i to d intersect its circumcircle respectively at B_i $(i = 1, 2, \cdots, n$. Prove that the sum

$$S = |A_1 B_1|^2 + \cdots + |A_n B_n|^2$$

is independent of d.

Solution. Select a system of coordinates so that O is the centre of the circumcircle and the $x-$axis (or real axis) is orthogonal to d. Without loss of generality, we may assume that the radius of the circumcircle is of length 1. Let a_k the the affix (complex number representative) of A_k $(1 \leq k \leq n)$. Then the a_k are solutions of the equation $z^n = \lambda$, where λ is a complex number with $|\lambda| = 1$. Since A_k and B_k are symmetrical with respect to the real axis, the affix of B_k is $\overline{a_k}$, the complex conjugate of a_k, for $1 \leq k \leq n$, Thus

$$A_k B_k^2 = |a_k - \overline{a_k}|^2 = (a_k - \overline{a_k})(\overline{a_k} - a_k) = 2a_k \overline{a_k} - a_k^2 - \overline{a_k}^2 = 2 - a_k - \overline{a_k}^2 \ .$$

Summing these inequalities yields that

$$\sum_{k=1}^{n} A_k B_k^2 = 2n - \sum_{k=1}^{n} a_k^2 - \sum_{k=1}^{n} \overline{a_k}^2 \ .$$

Since $\{a_k : 1 \leq k \leq n\}$ is a complete set of solutions of the equation $z^n = \lambda$, their sum and the sum of their pairwise products vanishes. Hence

$$0 = \sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} \overline{a_k}^2 \ .$$

Hence $\sum_{k=1}^{n} A_k B_k^2 = 2n$.