PROBLEMS FOR JUNE

Solutions should be submitted to
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no later than **July 31, 2000**.

Notes: The word *unique* means *exactly one*. A *regular octahedron* is a solid figure with eight faces, each of which is an equilateral triangle. You can think of gluing two square pyramids together along the square bases. The symbol $\lfloor u \rfloor$ denotes the greatest integer that does not exceed u.

13. Suppose that $x_1, x_2, \cdots, x_n$ are nonnegative real numbers for which $x_1 + x_2 + \cdots + x_n < \frac{1}{2}$. Prove that

$$(1 - x_1)(1 - x_2)\cdots(1 - x_n) > \frac{1}{2} ,$$

14. Given a convex quadrilateral, is it always possible to determine a point in its interior such that the four line segments joining the point to the midpoints of the sides divide the quadrilateral into four regions of equal area? If such a point exists, is it unique?

15. Determine all triples (x, y, z) of real numbers for which

$$x(y + 1) = y(z + 1) = z(x + 1) .$$

16. Suppose that $ABCDEZ$ is a regular octahedron whose pairs of opposite vertices are (A, Z), (B, D) and (C, E). The points F, G, H are chosen on the segments AB, AC, AD respectively such that $AF = AG = AH$.

    (a) Show that EF and DG must intersect in a point K, and that BG and EH must intersect in a point L.

    (b) Let EG meet the plane of AKL in M. Show that $AKML$ is a square.

17. Suppose that r is a real number. Define the sequence x_n recursively by $x_0 = 0$, $x_1 = 1$, $x_{n+2} = rx_{n+1} - x_n$ for $n \geq 0$. For which values of r is it true that

$$x_1 + x_3 + x_5 + \cdots + x_{2m-1} = x_m^2$$

    for $m = 1, 2, 3, 4, \cdots$.

18. Let a and b be integers. How many solutions in real pairs (x, y) does the system

$$\lfloor x \rfloor + 2y = a$$

$$\lfloor y \rfloor + 2x = b$$

    have?