International Mathematical Talent Search – Round 44

Problem 1/44. In a strange language there are only two letters, a and b, and it is postulated that the letter a is a word. Furthermore, all additional words are formed according to the following rules:

1. Given any word, a new word can be formed from it by adding a b at the right hand end.
2. If in any word a sequence aaa appears, a new word can be formed by replacing aaa by the letter b.
3. If in any word a sequence bbb appears, a new word can be formed by omitting bbb.
4. Given any word, a new word can be formed by writing down the sequence that constitutes the given word twice.

For example, by (4), aa is a word, and by (4) again, $aaaa$ is a word. Hence by (2) ba is a word, and by (1), bab ia also a word. Again, by (1), $babb$ is a word, and so by (4), $babbbabb$ is also a word. Finally, by (3) we find that $baabb$ is a word.

Prove that in this language $baabaabaa$ is not a word.

Problem 2/44. Let $f(x) = x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor$ for all positive real numbers x, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

1. Determine x so that $f(x) = 2001$.
2. Prove that $f(x) = 2002$ has no solutions.

Problem 3/44. Let f be a function defined on the set of integers, and assume that it satisfies the following properties:

1. $f(0) \neq 0$;
2. $f(1) = 3$; and
3. $f(x)f(y) = f(x + y) + f(x - y)$ for all integers x and y.

Determine $f(7)$.

Problem 4/44. A certain company has a faulty telephone system that sometimes transposes a pair of adjacent digits when someone dials a three-digit extension. Hence a call to $x318$ would ring at either $x318$, $x138$, or $x381$, while a call received at $x044$ would be intended for either $x404$ or $x044$. Rather than replace the system, the company is adding a computer to deduce which dialed extensions are in error and revert those numbers to their correct form. They have to leave out several possible extensions for this to work. What is the greatest number of three-digit extensions the company can assign under this plan?

Problem 5/44. Determine the smallest number of squares into which one can dissect a 11×13 rectangle and exhibit such a dissection. The squares need not be of different sizes, their bases should be integers, and they should not overlap.