International Mathematical Talent Search – Round 42

Problem 1/42. How many positive five-digit integers are there consisting of the digits $1, 2, 3, 4$, $5, 6, 7, 8, 9$, in which one digit appears once and two digits appear twice? For example, 41174 is one such number, while 75355 is not.

Problem 2/42. Determine, with proof, the positive integer whose square is exactly equal to the number

$$1 + \sum_{i=1}^{2001} (4i-2)^3.$$

Problem 3/42. Factor the expression

$$30(a^2+b^2+c^2+d^2) + 68ab - 75ac - 156ad - 61bc - 100bd + 87cd.$$

Problem 4/42. Let $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ be a 9-long vector of integers. Determine X if the following seven vectors were all obtained from X by deleting three of its components:

$Y_1 = (0,0,0,1,0,1)$, $Y_2 = (0,0,1,1,1,0)$, $Y_3 = (0,1,0,1,0,1)$,
$Y_4 = (1,0,0,0,1,1)$, $Y_5 = (1,0,1,1,1,1)$, $Y_6 = (1,1,1,1,0,1)$,
$Y_7 = (1,1,0,1,1,0)$.

Problem 5/42. Let R and S be points on the sides BC and AC, respectively, of $\triangle ABC$, and let P be the intersection of AR and BS. Determine the area of $\triangle ABC$ if the areas of $\triangle APS$, $\triangle APB$, and $\triangle BPR$ are $5, 6$, and 7, respectively.