International Mathematical Talent Search – Round 38

Problem 1/38. A well-known test for divisibility by 19 is as follows: Remove the last digit of the number, add twice that digit to the truncated number, and keep repeating this procedure until a number less than 20 is obtained. Then, the original number is divisible by 19 if and only if the final number is 19. The method is exemplified on the right; it is easy to check that indeed 67944 is divisible by 19, while 44976 is not.

$$
\begin{array}{r}
6\,7\,9\,4\,\not{4} \\
8 \\ \hline
6\,8\,0\,\not{2} \\
4 \\ \hline
6\,8\,\not{4} \\
8 \\ \hline
7\,\not{6} \\
1\,2 \\ \hline
1\,9
\end{array}
\qquad
\begin{array}{r}
4\,4\,9\,7\,\not{6} \\
1\,2 \\ \hline
4\,5\,0\,\not{9} \\
1\,8 \\ \hline
4\,6\,\not{8} \\
1\,6 \\ \hline
6\,\not{2} \\
4 \\ \hline
1\,0
\end{array}
$$

Find and prove a similar test for divisibility by 29.

Problem 2/38. Compute $1776^{1492!}$ (mod 2000); i.e., the remainder when $1776^{1492!}$ is divided by 2000. (As usual, the exclamation point denotes factorial.)

Problem 3/38. Given the arithmetic progression of integers

$$308, 973, 1638, 2303, 2968, 3633, 4298,$$

determine the unique geometric progression of integers,

$$b_1, b_2, b_3, b_4, b_5, b_6,$$

so that

$$308 < b_1 < 973 < b_2 < 1638 < b_3 < 2303 < b_4 < 2968 < b_5 < 3633 < b_6 < 4298.$$

Problem 4/38. Prove that every polyhedron has two vertices at which the same number of edges meet.

Problem 5/38. In $\triangle ABC$, segments $PQ, RS,$ and TU are parallel to sides $AB, BC,$ and CA, respectively, and intersect at the points $X, Y,$ and Z, as shown in the figure on the right.

Determine the area of $\triangle ABC$ if each of the segments $PQ, RS,$ and TU bisects (halves) the area of $\triangle ABC$, and if the area of $\triangle XYZ$ is one unit. Your answer should be in the form $a + b\sqrt{2}$, where a and b are positive integers.