International Mathematical Talent Search – Round 36

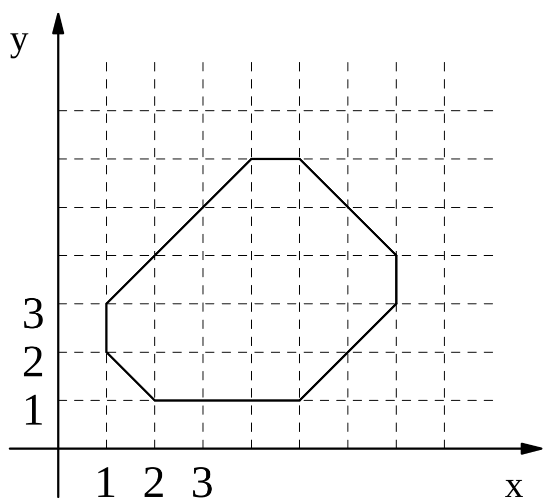
Problem 1/36. Determine the unique 9-digit integer M that has the following properties: (1) its digits are all distinct and non-zero; and (2) for every positive integer $m = 2, 3, 4, \ldots, 9$, the integer formed by the leftmost m digits of M is divisible by m.

Problem 2/36. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It is well–known that the sum of any 10 consecutive Fibonacci numbers is divisible by 11. Determine the smallest positive integer N so that the sum of any N consecutive Fibonacci numbers is divisible by 12.

Problem 3/36. Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}.$$

Problem 4/36. We will say that an octogon is integral if it is equiangular, its vertices are lattice points (i.e., points with integer coordinates), and its area is an integer. For example, the figure on the right shows an integral octogon of area 21. Determine, with proof, the smallest positive integer K so that for every positive integer $k \geq K$, there is an integral octogon of area k.



Problem 5/36. Let P be a point interior to square $ABCD$ so that $PA = a, PB = b,, PC = c$, and $c^2 = a^2 + 2b^2$. Given only the lengths a, b, and c, and using only a compass and straightedge, construct a square congruent to square $ABCD$.