International Mathematical Talent Search – Round 32

Problem 1/32. Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

Problem 2/32. For a nonzero integer i, the exponent of 2 in the prime factorization of i is called $ord_2(i)$. For example, $ord_2(9) = 0$ since 9 is odd, and $ord_2(28) = 2$ since $28 = 2^2 \times 7$. The numbers $3^n - 1$ for $n = 1, 2, 3, \ldots$ are all even, so $ord_2(3^n - 1) \geq 1$ for $n > 0$.
a) For which positive integers n is $ord_2(3^n - 1) = 1$?
b) For which positive integers n is $ord_2(3^n - 1) = 2$?
c) For which positive integers n is $ord_2(3^n - 1) = 3$?
Prove your answers.

Problem 3/32. Let f be a polynomial of degree 98, such that $f(k) = \frac{1}{k}$ for $k = 1, 2, 3, \ldots, 99$. Determine $f(100)$.

Problem 4/32. Let A consist of 16 elements of the set $\{1, 2, 3, \ldots, 106\}$, so that no two elements of A differ by $6, 9, 12, 15, 18,$ or 21. Prove that two of the elements of A must differ by 3.

Problem 5/32. In $\triangle ABC$, let D, E, and F be the midpoints of the sides of the triangle, and let P, Q, and R be the midpoints of the corresponding medians, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively, as shown in the figure below. Prove that the value of

$$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2}.$$

does not depend on the shape of $\triangle ABC$ and find that value.