International Mathematical Talent Search – Round 26

Problem 1/26. Assume that x, y, and z are positive real numbers that satisfy the equations given on the right.

$$\begin{aligned} x + y + xy &= 8, \\ y + z + yz &= 15, \\ z + x + zx &= 35. \end{aligned}$$

Determine the value of $x + y + z + xyz$.

Problem 2/26. Determine the number of non-similar regular polygons each of whose interior angles measures an integral number of degrees.

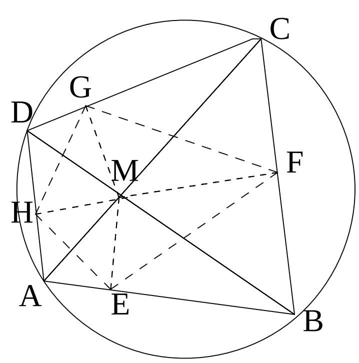
Problem 3/26. Substitute different digits $(0, 1, 2, \ldots, 9)$ for different letters in the alphametics on the right, so that the corresponding addition is correct, and the resulting value of M O N E Y is as large as possible. What is this value?

$$\begin{array}{cccccc} & & \mathrm{S} & \mathrm{H} & \mathrm{O} & \mathrm{W} \\ & & & & \mathrm{M} & \mathrm{E} \\ + & & & \mathrm{T} & \mathrm{H} & \mathrm{E} \\ \hline & \mathrm{M} & \mathrm{O} & \mathrm{N} & \mathrm{E} & \mathrm{Y} \end{array}$$

Problem 4/26. Prove that if $a \geq b \geq c > 0$, then

$$2a + 3b + 5c - \frac{8}{3}\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right) \leq \frac{1}{3}\left(\frac{a^2}{b} + \frac{b^2}{c} + 4\frac{c^2}{a}\right).$$

Problem 5/26. Let $ABCD$ be a convex quadrilateral inscribed in a circle, let M be the intersection point of the diagonals of $ABCD$, and let E, F, G, and H be the feet of the perpendiculars from M to the sides of $ABCD$, as shown in the figure on the right. Determine (with proof) the center of the circle inscribable in quadrilateral $EFGH$.



International Mathematical Talent Search – Round 27

Problem 1/27. Are there integers M, N, K, such that $M + N = K$ and

(i) each of them contains each of the seven digits $1, 2, 3, \ldots, 7$ exactly once?

(ii) each of them contains each of the nine digits $1, 2, 3, \ldots, 9$ exactly once?

Problem 2/27. Suppose that $R(n)$ counts the number of representations of the positive integer n as the sum of the squares of four non-negative integers, where we consider two representations equivalent if they differ only in the order of the summands. (For example, $R(7) = 1$ since $2^2 + 1^2 + 1^2 + 1^2$ is the only representation of 7 up to ordering.)

Prove that if k is a positive integer, then $R(2^k) + R(2^{k+1}) = 3$.

Problem 3/27. Assume that $f(1) = 0$, and that for all integers m and n,

$$f(m + n) = f(m) + f(n) + 3(4mn - 1).$$

Determine $f(19)$.

Problem 4/27. In the rectangular coordinate plane, $ABCD$ is a square, and $(31, 27), (42, 43), (60, 27)$, and $(46, 16)$ are points on its sides, AB, BC, CD, and DA, respectively. Determine the area of $ABCD$.

Problem 5/27. Is it possible to construct in the plane the midpoint of a given segment using compasses alone (i.e., without using a straight edge, except for drawing the segment)?

International Mathematical Talent Search – Round 28

Problem 1/28. For what integers b and c is $x = \sqrt{19} + \sqrt{98}$ a root of the equation $x^4 + bx^2 + c = 0$?

Problem 2/28. The sides of a triangle are of length $a, b,$ and c, where $a, b,$ and c are integers, $a > b$, and the angle opposite to c measures $60°$. Prove that a must be a composite number.

Problem 3/28. Determine, with a mathematical proof, the value of $\lfloor x \rfloor$; i.e., the greatest integer less than or equal to x, where

$$x = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{1,000,000}}.$$

Problem 4/28. Let n be a positive integer and assume that for each integer $k,\ 1 \le k \le n,$ we have two disks numbered k. It is desired to arrange the $2n$ disks in a row so that for each $k,\ 1 \le k \le n,$ there are k disks between the two disks that are numbered k. Prove that

(i) if $n = 6$, then no such arrangement is possible;

(ii) if $n = 7$, then it is possible to arrange the disks as desired.

Problem 5/28. Let S be the set of all points of a unit cube (i.e., a cube each of whose edges is of length 1) that are at least as far from any of the vertices of the cube as from the center of the cube. Determine the shape and volume of S.

International Mathematical Talent Search – Round 29

Problem 1/29. Several pairs of positive integers (m, n) satisfy the equation $19m + 90 + 8n = 1998$. Of these, $(100, 1)$ is the pair with the smallest value for n. Find the pair with the smallest value for m.

Problem 2/29. Determine the smallest rational number $\frac{r}{s}$ such that $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} \leq \frac{r}{s}$ whenever $k, m,$ and n are positive integers that satisfy the inequality $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.

Problem 3/29. It is possible to arrange eight of the nine numbers

$$2, 3, 4, 7, 10, 11, 12, 13, 15$$

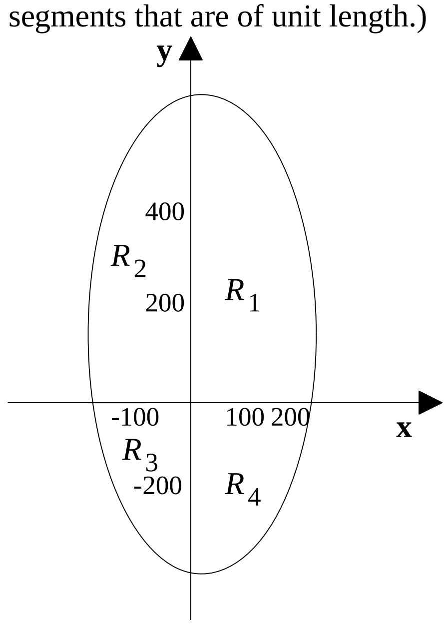
in the vacant squares of the 3 by 4 array shown on the right so that the arithmetic average of the numbers in each row and in each column is the same integer. Exhibit such an arrangement, and specify which one of the nine numbers must be left out when completing the array.

1			
	9		5
		14	

Problem 4/29. Show that it is possible to arrange seven distinct points in the plane so that among any three of these seven points, two of the three points are a unit distance apart. (Your solution should include a carefully prepared sketch of the seven points, along with all segments that are of unit length.)

Problem 5/29. The figure on the right shows the ellipse

$$\frac{(x-19)^2}{19} + \frac{(y-98)^2}{98} = 1998.$$



Let $R_1, R_2, R_3,$ and R_4 denote those areas within the ellipse that are in the 1st, 2nd, 3rd, and 4th quadrants, respectively. Determine the value of

$$R_1 - R_2 + R_3 - R_4.$$

International Mathematical Talent Search – Round 30

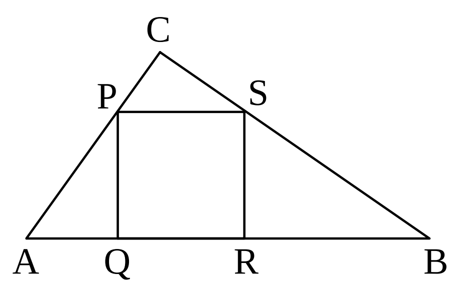
Problem 1/30. Determine the unique pair of real numbers (x, y) that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

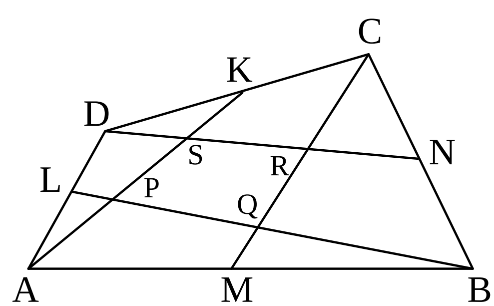
Problem 2/30. Prove that there are infinitely many ordered triples of positive integers (a, b, c) such that the greatest common divisor of $a, b,$ and c is 1, and the sum $a^2b^2 + b^2c^2 + c^2a^2$ is the square of an integer.

Problem 3/30. Nine cards can be numbered using positive half-integers $(1/2, 1, 3/2, 2, 5/2, \ldots)$ so that the sum of the numbers on a randomly chosen pair of cards gives an integer from 2 to 12 with the same frequency of occurence as rolling that sum on two standard dice. What are the numbers on the nine cards and how often does each number appear on the cards?

Problem 4/30. As shown in the figure on the right, square $PQRS$ is inscribed in right triangle ABC, whose right angle is at C, so that S and P are on sides BC and CA, respectively, while Q and R are on side AB. Prove that $AB \geq 3QR$ and determine when equality holds.



Problem 5/30. In the figure on the right, $ABCD$ is a convex quadrilateral, $K, L, M,$ and N are the midpoints of its sides, and $PQRS$ is the quadrilateral formed by the intersections of AK, BL, CM, and DN. Determine the area of quadrilateral $PQRS$ if the area of quadrilateral $ABCD$ is 3000, and the areas of quadrilaterals $AMQP$ and $CKSR$ are 513 and 388, respectively.



International Mathematical Talent Search – Round 31

Problem 1/31. Determine the three leftmost digits of the number

$$1^1 + 2^2 + 3^3 + \cdots + 999^{999} + 1000^{1000}.$$

Problem 2/31. There are infinitely many ordered pairs (m, n) of positive integers for which the sum

$$m + (m+1) + (m+2) + \cdots + (n-1) + n$$

is equal to the product mn. The four pairs with the smallest values of m are $(1, 1)$, $(3, 6)$, $(15, 35)$, and $(85, 204)$. Find three more (m, n) pairs.

Problem 3/31. The integers from 1 to 9 can be arranged into a 3×3 array so that the sum of the numbers in every row, column, and diagonal is a multiple of 9.

(a) Prove that the number in the center of the array must be a multiple of 3.

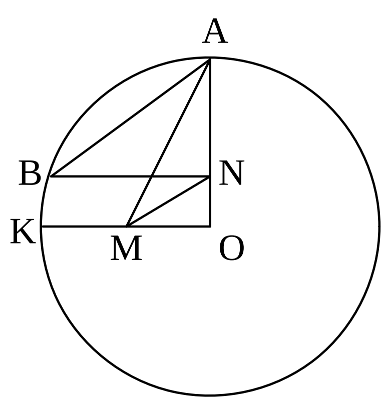
(b) Give an example of such an array with 6 in the center.

Problem 4/31. Prove that if $0 < x < \pi/2$, then

$$\sec^6 x + \csc^6 x + (\sec^6 x)(\csc^6 x) \geq 80.$$

Problem 5/31. In the figure shown on the right, O is the center of the circle, OK and OA are perpendicular to one another, M is the midpoint of OK, BN is parallel to OK, and $\angle AMN = \angle NMO$.

Determine the measure of $\angle ABN$ in degrees.



International Mathematical Talent Search – Round 32

Problem 1/32. Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

Problem 2/32. For a nonzero integer i, the exponent of 2 in the prime factorization of i is called $ord_2(i)$. For example, $ord_2(9) = 0$ since 9 is odd, and $ord_2(28) = 2$ since $28 = 2^2 \times 7$. The numbers $3^n - 1$ for $n = 1, 2, 3, \ldots$ are all even, so $ord_2(3^n - 1) \geq 1$ for $n > 0$.
a) For which positive integers n is $ord_2(3^n - 1) = 1$?
b) For which positive integers n is $ord_2(3^n - 1) = 2$?
c) For which positive integers n is $ord_2(3^n - 1) = 3$?
Prove your answers.

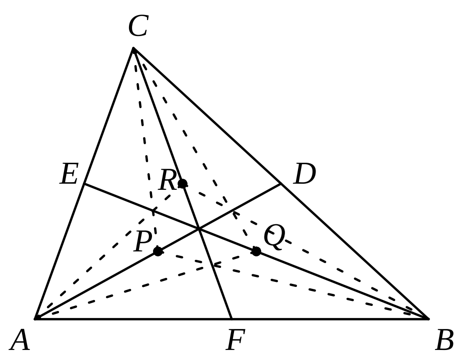
Problem 3/32. Let f be a polynomial of degree 98, such that $f(k) = \frac{1}{k}$ for $k = 1, 2, 3, \ldots, 99$. Determine $f(100)$.

Problem 4/32. Let A consist of 16 elements of the set $\{1, 2, 3, \ldots, 106\}$, so that no two elements of A differ by $6, 9, 12, 15, 18,$ or 21. Prove that two of the elements of A must differ by 3.

Problem 5/32. In $\triangle ABC$, let D, E, and F be the midpoints of the sides of the triangle, and let $P, Q,$ and R be the midpoints of the corresponding medians, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively, as shown in the figure below. Prove that the value of

$$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2}.$$

does not depend on the shape of $\triangle ABC$ and find that value.



International Mathematical Talent Search – Round 33

Problem 1/33. The digits of the three-digit integers a, b, and c are the nine non-zero digits $1, 2, 3, \ldots, 9$, each of them appearing exactly once. Given that the ratio $a : b : c$ is $1 : 3 : 5$, determine a, b, and c.

Problem 2/33. Let $N = 111\ldots1222\ldots2$, where there are 1999 digits of 1 followed by 1999 digits of 2. Express N as the product of four integers, each of them greater than 1.

Problem 3/33. Triangle ABC has angle A measuring $30°$, angle B measuring $60°$, and angle C measuring $90°$. Show four different ways to divide triangle ABC into four triangles, each similar to triangle ABC but with one quarter of the area. Prove that the angles and sizes of the smaller triangles are correct.

Problem 4/33. There are 8436 steel balls, each with radius 1 centimeter, stacked in a tetrahedral pile, with one ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the fourth, and so on. Determine the height ot the pile in centimeters.

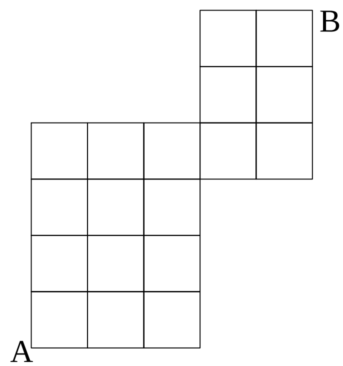
Problem 5/33. In a convex pentagon $ABCDE$ the sides have lengths 1, 2, 3, 4, and 5, though not necessarily in that order. Let F, G, H, and I be the midpoints of sides AB, BC, CD, and DE, respectively. Let X be the midpoint of segment FH, and Y be the midpoint of segment GI. The length of segment XY is an integer. Find all possible values of the length of side AE.

International Mathematical Talent Search – Round 34

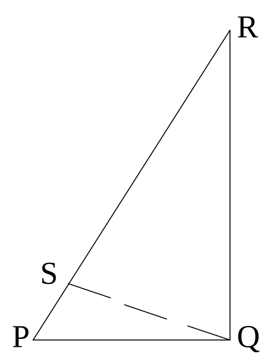
Problem 1/34. The number N consists of 1999 digits such that if each pair of consecutive digits in N were viewed as a two-digit number, then that number would either be a multiple of 17 or a multiple of 23. The sum of the digits of N is 9599. Determine the rightmost ten digits of N.

Problem 2/34. Let $\mathcal{C}$ be the set of non-negative integers which can be expressed as $1999s + 2000t$ where s and t are also non-negative integers.
(a) Show that $3,994,001$ is not in $\mathcal{C}$.
(b) Show that if $0 \leq n \leq 3,994,001$ and n is an integer not in $\mathcal{C}$, then $3,994,001 - n$ is in $\mathcal{C}$.

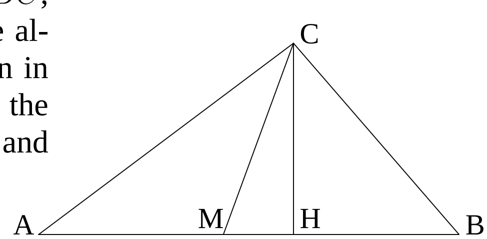
Problem 3/34. The figure on the right shows the map of Squareville, where each city block is of the same length. Two friends, Alexandra and Brianna, live at corners marked by A and B, respectively. They start walking toward each other's house, leaving at the same time, walking with the same speed, and independently choosing a path to the other's house with uniform distribution out of all possible minimum-distance paths (that is, all minimum-distance paths are equally likely). What is the probability that they will meet?



Problem 4/34. In $\triangle PQR$, $PQ = 8$, $QR = 13$, and $RP = 15$. Prove that there is a point S on the line segment $\overline{PR}$, but not at its endpoints, such that PS and QS are also integers.



Problem 5/34. In $\triangle ABC$, $AC > BC$, CM is the median, and CH is the altitude emanating from C, as shown in the figure on the right. Determine the measure of $\angle MCH$, if $\angle ACM$ and $\angle BCH$ each have measure 17°.



International Mathematical Talent Search – Round 35

Problem 1/35. We define the *repetition number* of a positive integer n to be the number of distinct digits of n when written in base 10. Prove that each positive integer has a multiple which has a repetition number less than or equal to 2.

Problem 2/35. Let a be a positive real number, n a positive integer, and define the *power tower* $a \uparrow n$ recursively with $a \uparrow 1 = a$, $a \uparrow (i+1) = a^{a \uparrow i}$ for $i = 1, 2, \ldots$. For example, we have $4 \uparrow 3 = 4^{4^4} = 4^{256}$, a number which has 155 digits. For each positive integer k, let x_k denote the unique positive real number solution of the equation $x \uparrow k = 10 \uparrow (k+1)$. Which is larger: x_{42} or x_{43}?

Problem 3/35. Suppose that the 32 computers in a certain network are numbered with the 5-bit integers $00000, 00001, \ldots, 11111$ (bit is short for binary digit). Suppose that there is a one-way connection from computer A to computer B if and only if A and B share four of their bits with the remaining bit being a 0 at A and a 1 at B. (For example, 10101 can send messages to 11101 and to 10111.) We say that a computer is at level k in the network if it has exactly k 1's in its label ($k = 0, 1, 2, \ldots 5$). Suppose further that we know that 12 computers, three at each of the levels 1, 2, 3, and 4, are malfunctioning, but we do not know which ones. Can we still be sure that we can send a message from 00000 to 11111?

Problem 4/35. We say that a triangle in the coordinate plane is *integral* if its three vertices have integer coordinates and if its three sides have integer lengths.

(a) Find an integral triangle with a perimeter of 42.
(b) Is there an integral triangle with a perimeter of 43?

Problem 5/35. We say that a finite set of points is *well scattered* on the surface of a sphere if every open hemisphere (half the surface of the sphere without its boundary) contains at least one of the points. While $\{(1,0,0), (0,1,0), (0,0,1)\}$ is not well scattered on the unit sphere (the sphere of radius 1 centered at the origin), but if you add the correct point P, it becomes well scattered. Find, with proof, all possible points P that would make the set well scattered.

International Mathematical Talent Search – Round 36

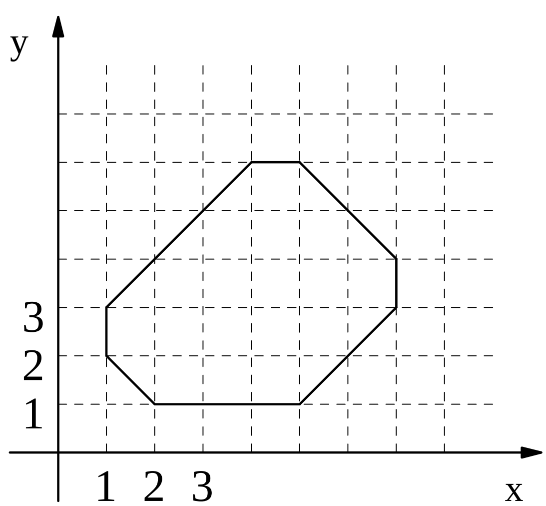
Problem 1/36. Determine the unique 9-digit integer M that has the following properties: (1) its digits are all distinct and non-zero; and (2) for every positive integer $m = 2, 3, 4, \ldots, 9$, the integer formed by the leftmost m digits of M is divisible by m.

Problem 2/36. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It is well–known that the sum of any 10 consecutive Fibonacci numbers is divisible by 11. Determine the smallest positive integer N so that the sum of any N consecutive Fibonacci numbers is divisible by 12.

Problem 3/36. Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}.$$

Problem 4/36. We will say that an octogon is integral if it is equiangular, its vertices are lattice points (i.e., points with integer coordinates), and its area is an integer. For example, the figure on the right shows an integral octogon of area 21. Determine, with proof, the smallest positive integer K so that for every positive integer $k \geq K$, there is an integral octogon of area k.



Problem 5/36. Let P be a point interior to square $ABCD$ so that $PA = a, PB = b,, PC = c$, and $c^2 = a^2 + 2b^2$. Given only the lengths $a, b,$ and c, and using only a compass and straightedge, construct a square congruent to square $ABCD$.

International Mathematical Talent Search – Round 37

Problem 1/37. Determine the smallest five-digit positive integer N such that $2N$ is also a five-digit integer and all ten digits from 0 to 9 are found in N and $2N$.

Problem 2/37. It was recently shown that $2^{2^{24}}+1$ is not a prime number. Find the four rightmost digits of this number.

Problem 3/37. Determine the integers a, b, c, d, and e for which

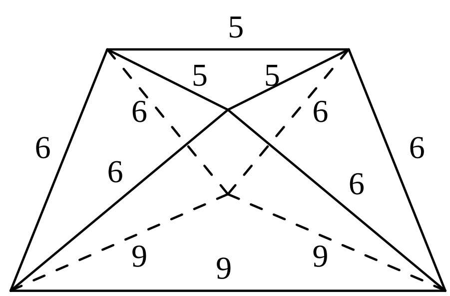
$$(x^2+ax+b)(x^3+cx^2+dx+e) = x^5-9x-27.$$

Problem 4/37. A sequence of real numbers $s_0, s_1, s_2, \ldots$ has the property that

$$\begin{aligned} s_i s_j &= s_{i+j}+s_{i-j} \quad \text{for all nonnegative integers } i \text{ and } j \text{ with } i \geq j, \\ s_i &= s_{i+12} \quad \text{for all nonnegative integers } i, \text{ and} \\ s_0 &> s_1 > s_2 > 0. \end{aligned}$$

Find the three numbers s_0, s_1, and s_2.

Problem 5/37. In the octahedron shown on the right, the base and top faces are equilateral triangles with sides measuring 9 and 5 units, and the lateral edges are all of length 6 units. Determine the height of the octahedron; i.e., the distance between the base and the top face.



International Mathematical Talent Search – Round 38

Problem 1/38. A well-known test for divisibility by 19 is as follows: Remove the last digit of the number, add twice that digit to the truncated number, and keep repeating this procedure until a number less than 20 is obtained. Then, the original number is divisible by 19 if and only if the final number is 19. The method is exemplified on the right; it is easy to check that indeed 67944 is divisible by 19, while 44976 is not.

$$\begin{array}{r} 6794\cancel{4} \\ 8 \\ \hline 680\cancel{2} \\ 4 \\ \hline 68\cancel{4} \\ 8 \\ \hline 7\cancel{6} \\ 12 \\ \hline 19 \end{array} \qquad \begin{array}{r} 4497\cancel{6} \\ 12 \\ \hline 450\cancel{9} \\ 18 \\ \hline 46\cancel{8} \\ 16 \\ \hline 6\cancel{2} \\ 4 \\ \hline 10 \end{array}$$

Find and prove a similar test for divisibility by 29.

Problem 2/38. Compute $1776^{1492!} \pmod{2000}$; i.e., the remainder when $1776^{1492!}$ is divided by 2000. (As usual, the exclamation point denotes factorial.)

Problem 3/38. Given the arithmetic progression of integers

$$308, 973, 1638, 2303, 2968, 3633, 4298,$$

determine the unique geometric progression of integers,

$$b_1, b_2, b_3, b_4, b_5, b_6,$$

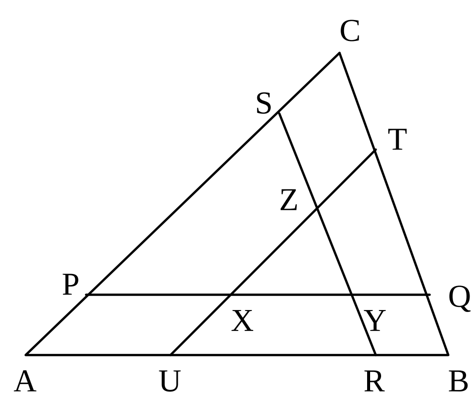
so that

$$308 < b_1 < 973 < b_2 < 1638 < b_3 < 2303 < b_4 < 2968 < b_5 < 3633 < b_6 < 4298.$$

Problem 4/38. Prove that every polyhedron has two vertices at which the same number of edges meet.

Problem 5/38. In $\triangle ABC$, segments $PQ, RS,$ and TU are parallel to sides $AB, BC,$ and CA, respectively, and intersect at the points $X, Y,$ and Z, as shown in the figure on the right.

Determine the area of $\triangle ABC$ if each of the segments $PQ, RS,$ and TU bisects (halves) the area of $\triangle ABC$, and if the area of $\triangle XYZ$ is one unit. Your answer should be in the form $a + b\sqrt{2}$, where a and b are positive integers.



International Mathematical Talent Search – Round 39

Problem 1/39. Find the smallest positive integer with the property that it has divisors ending in every decimal digit; i.e., divisors ending in $0, 1, 2, \ldots, 9$.

Problem 2/39. Assume that the irreducible fractions between 0 and 1, with denominators at most 99, are listed in ascending order. Determine which two fractions are adjacent to $\frac{17}{76}$ in this listing.

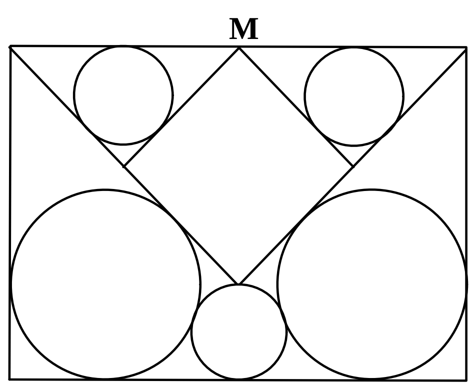
Problem 3/39. Let $p(x) = x^5 + x^2 + 1$ have roots r_1, r_2, r_3, r_4, r_5. Let $q(x) = x^2 - 2$. Determine the product $q(r_1)q(r_2)q(r_3)q(r_4)q(r_5)$.

Problem 4/39. Assume that each member of the sequence $< \diamond_i >_{i=1}^{\infty}$ is either a $+$ or $-$ sign. Determine the appropriate sequence of $+$ and $-$ signs so that

$$2 = \sqrt{6 \diamond_1 \sqrt{6 \diamond_2 \sqrt{6 \diamond_3 \cdots}}}.$$

Also determine what sequence of signs is necessary if the sixes in the nested roots are replaced by sevens. List all integers that work in the place of the sixes and the sequence of signs that are needed with them.

Problem 5/39. Three isosceles right triangles are erected from the larger side of a rectangle into the interior of the rectangle, as shown on the right, where M is the midpoint of that side. Five circles are inscribed tangent to some of the sides and to one another as shown. One of the circles touches the vertex of the largest triangle.



Find the ratios among the radii of the five circles.

International Mathematical Talent Search – Round 40

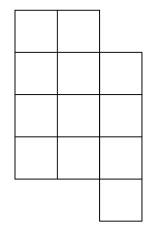
Problem 1/40. Determine all positive integers with the property that they are one more than the sum of the squares of their digits in base 10.

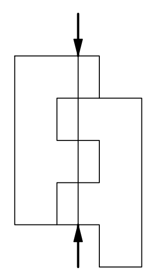
Problem 2/40. Prove that if n is an odd positive integer, then

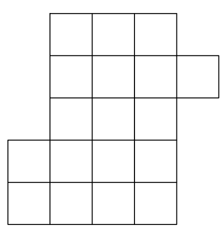
$$N = 2269^n + 1779^n + 1730^n - 1776^n$$

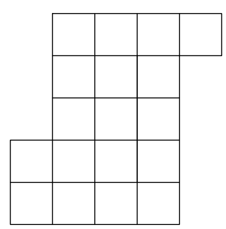
is an integer multiple of 2001.

Problem 3/40. The figure on the right can be divided into two congruent halves that are related to each other by a glide reflection, as shown below it. A glide reflection reflects a figure about a line, but also moves the reflected figure in a direction parallel to that line. For a square-grid figure, the only lines of reflection that keep its reflection on the grid are horizontal, vertical, $45°$ diagonal, and $135°$ diagonal. Of the two figures below, divide one figure into two congruent halves related by a glide reflection, and tell why the other figure cannot be divided like that.



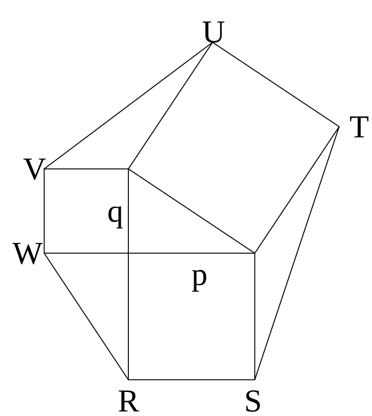






Problem 4/40. Let A and B be points on a circle which are not diametrically opposite, and let C be the midpoint of the smaller arc between A and B. Let D, E and F be the points determined by the intersections of the tangent lines to the circle at A, B, and C. Prove that the area of $\triangle DEF$ is greater than half of the area of $\triangle ABC$.

Problem 5/40. Hexagon $RSTUVW$ is constructed by starting with a right triangle of legs measuring p and q, constructing squares outwardly on the sides of this triangle, and then connecting the outer vertices of the squares, as shown in the figure on the right.



Given that p and q are integers with $p > q$, and that the area of $RSTUVW$ is 1922, determine p and q.

International Mathematical Talent Search – Round 41

Problem 1/41. Determine the unique positive two-digit integers m and n for which the approximation $\dfrac{m}{n} = .2328767$ is accurate to the seven decimals; i.e., $0.2328767 \le m/n < 0.2328768$.

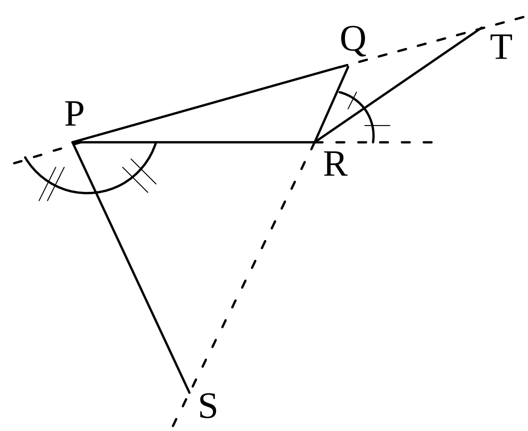
Problem 2/41. It is well known that there are infinitely many triples of integers (a, b, c) whose greatest common divisor is 1 and which satisfy the equation $a^2 + b^2 = c^2$.

Prove that there are also infinitely many triples of integers (r, s, t) whose greatest common divisor is 1 and which satisfy the equation $(rs)^2 + (st)^2 = (tr)^2$.

Problem 3/41. Suppose $\dfrac{\cos 3x}{\cos x} = \dfrac{1}{3}$ for some angle x, $0 \le x \le \dfrac{\pi}{2}$. Determine $\dfrac{\sin 3x}{\sin x}$ for the same x.

Problem 4/41. The projective plane of order three consists of 13 points and 13 lines. These lines are not Euclidean straight lines; instead they are sets of four points with the properties that each pair of lines has exactly one point in common and each pair of points has exactly one line that contains both points. Suppose the points are labeled 1 through 13 and six of the lines are $A = \{1, 2, 4, 8\}$, $B = \{1, 3, 5, 9\}$, $C = \{2, 3, 6, 10\}$, $D = \{4, 5, 10, 11\}$, $E = \{4, 6, 9, 12\}$, and $F = \{5, 6, 8, 13\}$. What is the line that contains 7 and 8?

Problem 5/41. In $\triangle PQR$, $QR < PR < PQ$ so that the exterior angle bisector through P intersects ray $\overrightarrow{QR}$ at point S, and the exterior angle bisector at R intersects ray $\overrightarrow{PQ}$ at point T, as shown on the right. Given that $PR = PS = RT$, determine, with proof, the measure of $\angle PRQ$.



International Mathematical Talent Search – Round 42

Problem 1/42. How many positive five-digit integers are there consisting of the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$, in which one digit appears once and two digits appear twice? For example, 41174 is one such number, while 75355 is not.

Problem 2/42. Determine, with proof, the positive integer whose square is exactly equal to the number

$$1 + \sum_{i=1}^{2001} (4i - 2)^3.$$

Problem 3/42. Factor the expression

$$30(a^2 + b^2 + c^2 + d^2) + 68ab - 75ac - 156ad - 61bc - 100bd + 87cd.$$

Problem 4/42. Let $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ be a 9-long vector of integers. Determine X if the following seven vectors were all obtained from X by deleting three of its components:

$$Y_1 = (0,0,0,1,0,1), \quad Y_2 = (0,0,1,1,1,0), \quad Y_3 = (0,1,0,1,0,1),$$
$$Y_4 = (1,0,0,0,1,1), \quad Y_5 = (1,0,1,1,1,1), \quad Y_6 = (1,1,1,1,0,1),$$
$$Y_7 = (1,1,0,1,1,0).$$

Problem 5/42. Let R and S be points on the sides BC and AC, respectively, of $\triangle ABC$, and let P be the intersection of AR and BS. Determine the area of $\triangle ABC$ if the areas of $\triangle APS$, $\triangle APB$, and $\triangle BPR$ are $5, 6$, and 7, respectively.

International Mathematical Talent Search – Round 43

Problem 1/43. We will say that a rearrangement of the letters of a word *has no fixed letters* if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, the blocks of letters below shows that $E\,S\,A\,R\,E\,T$ is a rearrangement with no fixed letters of $T\,E\,R\,E\,S\,A$, but $R\,E\,A\,S\,T\,E$ is not.

$$\begin{array}{cccccc} T & E & R & E & S & A \\ E & S & A & R & E & T \end{array} \qquad\qquad \begin{array}{cccccc} T & E & R & E & S & A \\ R & E & A & S & T & E \end{array}$$

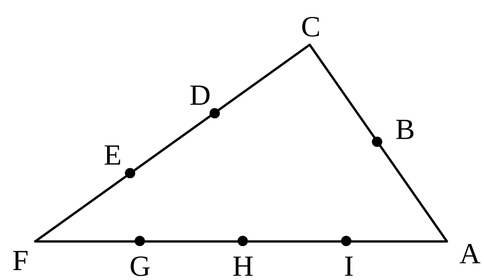
How many distinguishable rearrangements with no fixed letters does $T\,E\,R\,E\,S\,A$ have? (The two Es are considered identical.)

Problem 2/43. Find five different sets of three positive integers $\{k, m, n\}$, such that $k < m < n$ and

$$\frac{1}{k} + \frac{1}{m} + \frac{1}{n} = \frac{19}{84}.$$

Problem 3/43. Suppose $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients and suppose $(p(x))^2$ is a polynomial all of whose coefficients are non-negative. Is it necessarily true that all the coefficients of $p(x)$ must be non-negative? Justify your answer.

Problem 4/43. As shown in the figure on the right, in $\triangle ACF$, B is the midpoint of $\overline{AC}$, D and E divide side $\overline{CF}$ into three equal parts, while G, H and I divide side $\overline{FA}$ into four equal parts.



Seventeen segments are drawn to connect these six points to one another and to the opposite vertices of the triangle. Determine the points interior to $\triangle ACF$ at which three or more of these line segments intersect one another.

Problem 5/43. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, spaced 14 units apart, measured along the straight line connecting them. If the radii of the circles are 18 and 25 units, what is the radius of the sphere?

International Mathematical Talent Search – Round 44

Problem 1/44. In a strange language there are only two letters, a and b, and it is postulated that the letter a is a word. Furthermore, all additional words are formed according to the following rules:

1. Given any word, a new word can be formed from it by adding a b at the right hand end.
2. If in any word a sequence aaa appears, a new word can be formed by replacing aaa by the letter b.
3. If in any word a sequence bbb appears, a new word can be formed by omitting bbb.
4. Given any word, a new word can be formed by writing down the sequence that constitutes the given word twice.

For example, by (4), aa is a word, and by (4) again, $aaaa$ is a word. Hence by (2) ba is a word, and by (1), bab ia also a word. Again, by (1), $babb$ is a word, and so by (4), $babbbabb$ is also a word. Finally, by (3) we find that $baabb$ is a word.

Prove that in this language $baabaabaa$ is not a word.

Problem 2/44. Let $f(x) = x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor$ for all positive real numbers x, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

1. Determine x so that $f(x) = 2001$.
2. Prove that $f(x) = 2002$ has no solutions.

Problem 3/44. Let f be a function defined on the set of integers, and assume that it satisfies the following properties:

1. $f(0) \neq 0$;
2. $f(1) = 3$; and
3. $f(x)f(y) = f(x+y) + f(x-y)$ for all integers x and y.

Determine $f(7)$.

Problem 4/44. A certain company has a faulty telephone system that sometimes transposes a pair of adjacent digits when someone dials a three-digit extension. Hence a call to $x318$ would ring at either $x318$, $x138$, or $x381$, while a call received at $x044$ would be intended for either $x404$ or $x044$. Rather than replace the system, the company is adding a computer to deduce which dialed extensions are in error and revert those numbers to their correct form. They have to leave out several possible extensions for this to work. What is the greatest number of three-digit extensions the company can assign under this plan?

Problem 5/44. Determine the smallest number of squares into which one can dissect a 11×13 rectangle and exhibit such a dissection. The squares need not be of different sizes, their bases should be integers, and they should not overlap.